PMM U.S.S.R.,Vol.53,No.4,pp.417-424,1989
Printed in Great Britain

0021-8928/89 $10.00+0.00

THE PRESERVATION OF THE STABILITY OF A MECHANICAL SYSTEM WHEN A NON-RETAINING CONSTRAINT IS WEAKENED*

A.P. IVANOV

A system with ideal non-retaining constraint $q_1 \geqslant 0$ is studied in the neighbourhood of an equilibrium position in which the reaction of the constraint is non-zero. It is assumed that the equilibrium is stable when $q_1 \equiv 0$. A family of periodic motions with impacts on the constraint is shown to exist, the period of whose motions tends to zero along with their amplitude. The orbital stability of the periodic motions is studied to a first approximation. It is shown that certain results of KAM theory can be used for non-linear analysis. In particular, conditions are obtained for the equilibrium position of a multidimensional system to be stable for a majority of initial conditions.

1. *Purpose of the investigation.* We consider a canonical system M with the Hamiltonian

$$H = T + \Pi,\ 2T = a_{ij}(\mathbf{q})\, p_i p_j,\ \Pi = \Pi(\mathbf{q}),\ \mathbf{q} \in R^n \tag{1.1}$$

and the ideal non-retaining constraint $q_1 \geqslant 0$; impacts on the constraint are absolutely elastic. Here and throughout, we understand summation over repeated subscripts: $i, j = 1, \ldots, n$; $k, m = 2, \ldots, n$.

If, when $\mathbf{q} = 0$ we have

$$\partial\Pi/\partial q_1 > 0,\ \ \partial\Pi/\partial q_2 = 0, \ldots, \partial\Pi/\partial q_n = 0 \tag{1.2}$$

then the origin of the phase space is the equilibrium position of M /1/.

The stability of the equilibrium position can be determined by considering the system M' with $n-1$ degrees of freedom and Hamiltonian $H' = H\,|_{q_1=p_1=0}$. We have /2/

Theorem 1. We write the function $\Pi' = \Pi\,|_{q_1=0}$ as

$$\Pi' = \Pi_s' + \Pi_{s+1}' + \ldots \tag{1.3}$$

where Π_s' is a homogeneous polynomial of $q_2, \ldots, q_n$ of degree s. If Π_s' is positive definite (it can take negative values), the zero equilibrium position of system (1.1) is Lyapunov stable (unstable).

The connection between the stability of systems M and M' is more complicated when terms $a_j p_j$ are present in H (the equilibrium conditions then include, in addition to Eqs.(1.2), the requirements $a_j = 0$ for $\mathbf{q} = 0$).

The stability of system M' is necessary but not sufficient for M to be stable, see /2/, as shown by the following example.

Example. Consider the system with Hamiltonian

$$H = p_1^2/2 + |q_1| + r_2 - 2r_3 + (p_1^2/2 + |q_1|)\, r_2 r_3^{1/2} \sin(2\varphi_2 + \varphi_3)$$
$$2r_j = q_j^2 + p_j^2 \quad (j = 2, 3)$$

where r, φ are canonical polar variables /3/, which describe the motion in the neighbourhood of equilibrium. The system M' is described by the Hamiltonian $H' = r_2 - 2r_3$ and is integrable: $r_2 = \text{const}$, $r_3 = \text{const}$. In view of this, the equilibrium position $r_2 = r_3 = 0$ of system M' is stable, in spite of the fact that the Hamiltonian of the disturbed motion changes sign (gyro-stabilization). If, by virtue of the initial disturbances, the values of q_1, p_1 are non-zero, then, since system M has the first integral $p_1^2/2 + |q_1| = \text{const}$, we obtain the well-known case of instability with third-order resonance /3/.

The sufficient condition for stability are given by the following theorem /2/.

Theorem 2. If the function Π_s' in expansion (1.3) is positive definite, the stability of system M (with the presence of terms, linear in $\mathbf{p}$) is Lyapunov stable.

Note. If the system M' has one degree of freedom, its stability may not be ensured by gyrostabilization, and is only possible with zero degree of instability. Here, therefore,

---

*Prikl. Matem. Mekhan., 53,4,539-548,1989

Theorem 1 holds even when terms linear in $\mathbf{p}$ are present in (1.1).

In the present paper we study the behaviour of system M in the neighbourhood of the origin of phase space when the system M' is stable. Apart from examining the equilibrium in cases which cannot be decided by Theorems 1 and 2, we study the motion with periodic impacts and their orbital stability.

2. The Hamiltonian of the disturbed motion. Let us find the auxiliary system M^* with n degrees of freedom and without a non-retaining constraint. For this, we assume that the generalized coordinates in M are such that, in (1.1), $a_{1k} \equiv 0$ $(k = 2, \ldots, n)$. The existence of these coordinates is proved in /4/, and the corresponding replacement of the variables in the canonical form is quite simple. If the generating function of the transformation $\mathbf{q}, \mathbf{p} \mapsto \mathbf{Q}, \mathbf{P}$ is such that

$$Z = q_1 P_1 + f_k(\mathbf{q}) P_k \tag{2.1}$$

then $p_1 = P_1 + P_m \partial f_m / \partial q_1$, $p_k = P_m \partial f_m / \partial q_k$, and the condition that there are no products $P_1 P_j$ in the transformed Hamiltonian can be written as $\partial f_k / \partial q_1 = -a_{1k}/a_{11}$. On adding to these differential equations the initial conditions $f_k = q_k$ for $q_1 = 0$, we obtain the replacement (2.1), which, by Lindelöff's theorem /5/, is non-degenerate for $q_1 > 0$.

When there are terms linear in $\mathbf{p}$ in Hamiltonian (1.1), we can assume without loss of generality that, in addition to the equations $a_{1k} \equiv 0$, we have the condition $a_1 \equiv 0$ /2/.

The Hamilton function H^* is given by

$$H^*(\mathbf{q}, \mathbf{p}) = H(|q_1|, q_2, \ldots, q_n, \mathbf{p}), \quad \partial H/\partial q_1|_{q_1=0} = \min\{0, \partial H/\partial q_1|_{q_1=+0}\} \tag{2.2}$$

The trajectories $\mathbf{q}^*(t), \mathbf{p}^*(t)$ of system M^* are continuous and are connected with the trajectories of M by

$$q_1(t) = |q_1^*(t)|, \quad p_1(t) = p_1^*(t) \operatorname{sign} q_1^*(t), \quad q_k(t) = q_k^*(t), \; p_k = p_k^* \tag{2.3}$$

If conditions (1.2) hold in the equilibrium position $\mathbf{q} = \mathbf{p} = 0$, we can write H^* as

$$\begin{gathered} H^* = {}^1/_2 a_{11}^{\circ} p_1^2 + {}^1/_2 \overset{\circ}{a}_{km} p_k p_m + c_{km} q_k p_m + b_1 |q_1| + \\ |q_1| [U_1(\mathbf{q}') + V_1(\mathbf{p}')] + U_2(\mathbf{q}') + {}^1/_2 b_{11} q_1^2 + H_3 + \ldots + H_s + \ldots \\ b_1 > 0, \quad \mathbf{q}' = (q_2, \ldots, q_n), \quad \mathbf{p}' = (p_2, \ldots, p_n) \end{gathered} \tag{2.4}$$

where U_1 and V_1 are linear forms, and U_2 is a quadratic form, and H_s is a sum of terms of degree s in $|q_1|, \mathbf{q}', \mathbf{p}$.

Since we are assuming that the equilibrium of system M' is stable, there is a linear canonical transformation $\mathbf{q}', \mathbf{p}' \to \mathbf{Q}', \mathbf{P}'$ such that the part of the Hamiltonian (2.4) which is quadractic in Q', P' has the normal form /3/

$$ {}^1/_2 \overset{\circ}{a}_{km} p_k p_m + c_{km} q_k p_m + U_2(\mathbf{q}') = {}^1/_2 \lambda_k (\mathbf{Q}_k^2 + \mathbf{P}_k^2) \quad \lambda_k \in R \tag{2.5}$$

3. Periodic motions in the neighbourhood of equilibrium. For local non-linear analysis, an important role is played by the Lyapunov holomorphic integral theorem /6/, which says that, if the Hamilton function is analytic, then for every pair of pure imaginary roots $\pm i\lambda$ of the characteristic equation, and when there are no other roots $i\lambda N$, $N \in Z$, a family of periodic solutions exists, whose period tends to $2\pi/\lambda$ as the amplitude tends to zero.

The smoothness requirement for the function (2.4), (2.5) can be weakened, as is shown by the following theorem.

Theorem 3. In the neighbourhood of $\mathbf{q} = \mathbf{p} = 0$ let the Hamiltonian be

$$\begin{gathered} 2H = a_{11}^{\circ} p_1^2 + 2b_1 |q_1| + b_{11} q_1^2 + 2|q_1| (b_{1k} q_k + \\ c_{1k} p_k) + \lambda_k (q_k^2 + p_k^2) + H_1, \quad a_{11}^{\circ} > 0, \quad b_1 > 0, \quad \lambda_k \in R \end{gathered} \tag{3.1}$$

where $H_1 = H_1(\mathbf{q}, \mathbf{p})$ is a thrice continuously differentiable function which vanishes along with its first- and second-order partial derivatives for $\mathbf{q} = \mathbf{p} = 0$. If none of the λ_j is zero, system (3.1), for $0 < \varepsilon < \varepsilon_0$, has a family of ε-periodic solutions, for which

$$q_1 = O(\varepsilon^2), \quad p_1 = O(\varepsilon), \quad q_j, p_j = O(\varepsilon^s), \quad s \geqslant 2 \tag{3.2}$$

Note. The family (3.2) corresponds to motions of system M with impacts on the constraint. As distinct from the classical statement, the period of the motions tends to zero with their amplitude.

Proof of Theorem 3. We will change from variables q_1, p_1 to the canonical variables I, w by means of the replacement, 2π-periodic in w, of the form for $|w| \leqslant \pi$

$$q_1 = \alpha I^{2/3} w(\pi - |w|), \quad p_1 = \beta I^{1/3} (\pi/2 - |w|)$$

$$\alpha = 2\frac{a_{11}^\circ}{b_1}\left(\frac{3b_1}{2\pi^2 a_{11}^\circ}\right)^{2/3}, \quad \beta = 2\left(\frac{3b_1}{2\pi^2 a_{11}^\circ}\right)^{1/3}$$

and simultaneously introduce the canonical variables x_k, y_k by $q_k + ip_k = \sqrt{2}x_k$, $iq_k + p_k = \sqrt{2}y_k$.

In the new variables, (3.1) is

$$H = \gamma I^{3/2} + i\lambda_k x_k y_k + I^{3/4} L_1 + I^{3/2} F(I, w, \mathbf{x}, \mathbf{y}) + G(\mathbf{x}, \mathbf{y}),\ \gamma = {}^1/_4\pi^2 b_1\alpha \tag{3.3}$$

where L_1 is a linear function of $\mathbf{x}, \mathbf{y}$ with coefficients, 2π-periodic in w, and the function F is continuous and 2π-periodic in w, is continuous in I and continuously differentiable for $I \neq 0$, is twice continuously differentiable with respect to $\mathbf{x}, \mathbf{y}$, and for $\mathbf{x} = \mathbf{y} = 0$ we have $F = \partial F/\partial \mathbf{x} = \partial F/\partial \mathbf{y} = 0$, while the function G is thrice continuously differentiable and its first- and second-order partial derivatives vanish for $\mathbf{x} = \mathbf{y} = 0$.

If it turns out that $L_1 \equiv 0$, the periodic solutions in question can be written at once

$$\mathbf{x} = \mathbf{y} = 0,\ \ I = I_0 = \text{const},\ \ w^\cdot = {}^2/_3\gamma I_0^{-1/3} = 2\pi/\varepsilon \tag{3.4}$$

To obtain the solutions in the general case, some transformations are needed. If $I \sim \varepsilon^3$, $\mathbf{x}, \mathbf{y} \sim \varepsilon^{3/2}$, then H has the order ε^2. We fix $H = H^\circ \neq 0$ and put $I = (H^\circ/\gamma)^{2/3} + r$. Regarding Eq.(3.3) as an implicitly specified function $r(w, \mathbf{x}, \mathbf{y})$, we perform iso-energetic reduction /6/. The reduced system is

$$\begin{gathered} d\mathbf{x}/dw = \partial r/\partial \mathbf{y},\ d\mathbf{y}/dw = -\partial r/\partial \mathbf{x},\ r = i\alpha_1\varepsilon\lambda_k x_k y_k + \\ \varepsilon^3 [f_k(w) x_k + g_k(w) y_k] + \varepsilon^3 F_0(w, \varepsilon, \mathbf{x}, \mathbf{y}),\ \ \alpha_1 \in R \end{gathered} \tag{3.5}$$

where F_0 is twice continuously differentiable with respect to $\mathbf{x}, \mathbf{y}$, and $F_0 = \partial F_0/\partial \mathbf{x} = \partial F_0/\partial \mathbf{y} = 0$, $\mathbf{x} = \mathbf{y} = 0$.

In system (3.5) we perform the canonical transformation $\mathbf{x}, \mathbf{y} \to \mathbf{x}^*, \mathbf{y}^*$ with generating function $S = x_k y_k^* + a_k(w) x_k + b_k(w) y_k^*$. The connection between the old and new variables is given by $\mathbf{y} = \mathbf{y}^* + \mathbf{a}$, $\mathbf{x}^* = \mathbf{x} + \mathbf{b}$, and the new Hamiltonian is

$$r^* = r + \partial S/\partial w = r(\mathbf{x}^* - \mathbf{b}, \mathbf{y}^* + \mathbf{a}) + a_k'(x_k^* - b_k) + b_k' y_k^* \tag{3.6}$$

We choose the coefficients $a_k(w), b_k(w)$ in such a way that the first-order terms in $\mathbf{x}^*, \mathbf{y}^*$ in (3.6) vanish. Noting that the linear part of the function $r(\mathbf{x}^* - \mathbf{b}, \mathbf{y}^* + \mathbf{a})$ is given by $r_1 = x_k^* \partial r(-\mathbf{b}, \mathbf{a})/\partial x_k^* + y_k^* \partial r(-\mathbf{b}, \mathbf{a})/\partial y_k^*$, we obtain the following system of equations for $\mathbf{a}, \mathbf{b}$

$$\begin{gathered} a_j' + i\alpha_1\varepsilon\lambda_j a_j + \varepsilon^3 f_j + \varepsilon^3 \partial F_0(w, -\mathbf{b}, \mathbf{a})/\partial x_j = 0 \\ b_j' - \bar{i}\alpha_1\varepsilon\lambda_j b_j + \varepsilon^3 g_j + \varepsilon^3 \partial F_0(w, -\mathbf{b}, \mathbf{a})/\partial y_j = 0 \end{gathered} \tag{3.7}$$

We perform the task of normalization by finding the 2π-periodic solution of system (3.7). In this case, system (3.6) has the form $r^* = i\varepsilon\alpha_1\lambda_k x_k^* y_k^* + \varepsilon^3 F_0^*(w, \varepsilon, \mathbf{x}^*, \mathbf{y}^*)$, where the partial derivatives of F_0^* with respect to $\mathbf{x}^*, \mathbf{y}^*$ vanish for $\mathbf{x}^* = \mathbf{y}^* = 0$. The required periodic solutions are $\mathbf{x}^* = \mathbf{y}^* = 0,\ r^* = \varepsilon^3 F_0^*(w, \varepsilon, 0, 0)$.

We obtain the periodic solutions of system (3.7) by successive approximations. As a first approximation we take $\mathbf{a}^{(1)} = \mathbf{b}^{(1)} = 0$. One approximation is connected with the next by the relations

$$\begin{gathered} a_j'^{(s+1)} + i\varepsilon\alpha_1\lambda_j a_j^{(s+1)} + \varepsilon^3 f_j = -\varepsilon^3 \partial F_0^{(s)}/\partial x_j \\ b_j'^{(s+1)} - i\varepsilon\alpha_1\lambda_j b_j^{(s+1)} + \varepsilon^3 g_j = -\varepsilon^3 \partial F_0^{(s)}/\partial y_j \\ F_0^{(s)} = F_0(w, -\mathbf{b}^{(s)}, \mathbf{a}^{(s)}) \end{gathered} \tag{3.8}$$

System (3.8) splits up into $(2n - 2)$ linear equations of the same type $z' + i\varepsilon\lambda z = -\varepsilon^3 f(w)$, where by hypothesis $\lambda \neq 0$. For sufficiently small ε (so that $\varepsilon\lambda$ is not an integer), this equation has the periodic solution

$$\begin{gathered} z = -\varepsilon^3 e^{-i\varepsilon\lambda w} \int_0^w e^{i\varepsilon\lambda w} f(w)\, dw + Ce^{-i\varepsilon\lambda w} \\ C = \varepsilon^3 (1 - e^{2\pi i\varepsilon\lambda})^{-1} \int_0^{2\pi} e^{i\varepsilon\lambda w} f(w)\, dw \sim \varepsilon^3 \end{gathered} \tag{3.9}$$

The successive approximations are given as 2π-periodic functions of w by (3.8) and (3.9). If we define the norm of a vector function as

$$\|\mathbf{a}, \mathbf{b}\| = \max_{w \in [0, 2\pi]} \sum_{j=2}^{n} (|a_j(w)| + |b_j(w)|),$$

then we obtain, for sufficiently small ε, from the continuous differentiability of $\partial F_0/\partial \mathbf{x}$, $\partial F_0/\partial \mathbf{y}$ and (3.9), the inequality

$$\|\mathbf{a}_1^{(s+1)} - \mathbf{a}_2^{(s+1)}, \mathbf{b}_1^{(s+1)} - \mathbf{b}_2^{(s+1)}\| \leqslant \mu \|\mathbf{a}_1^{(s)} - \mathbf{a}_2^{(s)}, \mathbf{b}_1^{(s)} - \mathbf{b}_2^{(s)}\|, \quad \mu < 1 \tag{3.10}$$

where $\mathbf{a}_1^{(s)}\ \mathbf{b}_1^{(s)}$, $\mathbf{a}_2^{(s)}, \mathbf{b}_2^{(s)}$ are arbitrary periodic functions of w, and $\mathbf{a}_1^{(s+1)}$, $\mathbf{b}_1^{(s+1)}$, $\mathbf{a}_2^{(s+1)}$, $\mathbf{b}_2^{(s+1)}$ are the corresponding solutions of system (3.8).

In view of (3.10), the iterative process converges to the solution of system (3.7). This completes the construction of the periodic solutions. Estimate (3.2) follows from (3.9).

Corollary /7, 8/. Function (3.1) can be reduced by canonical transformation to the form

$$H = f(I) + F(w, I, \mathbf{q}, \mathbf{p}) \tag{3.11}$$

where $\partial F/\partial \mathbf{q} = \partial F/\partial \mathbf{p} = 0$ for $\mathbf{q} = \mathbf{p} = 0$, and $f' \to \infty$ as $I \to 0$.

Note. If H_1 in (3.1) is analytic, then the solution of system (3.7) depends analytically on ε /9/. The periodic solutions (3.2) can, in this case, be written as convergent series in the parameter ε, as in the classical statement of the theorem.

Example. The motion of a particle along a vertical above a horizontal support is given by the Hamiltonian $H = {}^1/_2 p^2 + g|q|$, where g is the acceleration due to gravity. The general solution can be expressed by ε-periodic functions, which have the form, when $|t| \leqslant \varepsilon/2$

$$q = {}^1/_2 gt\,({}^1/_2\varepsilon - |t|), \quad p = g\,({}^1/_4\varepsilon - |t|)$$

Here, $\max|q| = q({}^1/_4\varepsilon) = {}^1/_{32} g\varepsilon^2$, $\max|p| = p(0) = {}^1/_4 g\varepsilon$.

4. Orbital stability of periodic motions. When studying the orbital stability of the periodic motions obtained above, a primary role is played by the quadratic part of functions (3.6)

$$r^* = {}^1/_2 \alpha_1 \varepsilon \lambda_k (q_k^2 + p_k^2) + \varepsilon^3 B(w, \varepsilon, \mathbf{q}, \mathbf{p}) \tag{4.1}$$

where B is a quadratic form in $\mathbf{q}, \mathbf{p}$ with coefficients, 2π-periodic in w. The first term in (4.1) is the principal part as $\varepsilon \to 0$.

To see how the multiplicators of system (4.1) depend on the parameter ε (this dependence characterizes the amplitude of the periodic solutions) we use the Krein-Gel'fand-Lidskii theorem on strong stability /10/. By this theorem, if the numbers σ_j do not satisfy a resonance relation

$$\sigma_r + \sigma_s = N, \quad N \in Z \quad (r, s = 1, \ldots, n) \tag{4.2}$$

then there is a number $\delta = \delta(\sigma) > 0$ such that, when we have

$$\max_{t \in [0, 2\pi],\ |\mathbf{q}|=1,\ |\mathbf{p}|=1} |H_1(t, \mathbf{q}, \mathbf{p})| < \delta \tag{4.3}$$

the system with Hamiltonian

$$H = {}^1/_2 \sigma_k (q_k^2 + p_k^2) + H_1(t, \mathbf{q}, \mathbf{p}), \quad \mathbf{q}, \mathbf{p} \in R^n \tag{4.4}$$

is stable. In other words, when there are no resonances, the multiplicators when the Hamilton function is slightly disturbed will not leave the unit circle in the complex plane.

As applied to system (4.1), if none of the λ_k is zero or negative, then the condition for there to be no resonances of type (4.2) can be satisfied for $\varepsilon \in]0, \varepsilon_0]$, where ε_0 is given by

$$\alpha_1 \varepsilon_0 \max_j |\lambda_j| < {}^1/_2, \quad \alpha_1 \varepsilon_0 \max_{r \neq s} |\lambda_r + \lambda_s| < 1 \tag{4.5}$$

Along with (4.1), we take the system dependent on the parameter μ

$$H_\mu = \alpha_1 \mu \lambda_k (q_k^2 + p_k^2)/2 + H_1(w, \varepsilon, \mathbf{q}, \mathbf{p}) \tag{4.6}$$

If $\mu \in]0, \varepsilon_0]$, there is a number $\delta = \delta(\mu) > 0$ such that, when (4.3) holds, system (4.6) is stable, i.e., all its solutions $\mathbf{q}_\mu(w)$, $\mathbf{p}_\mu(w)$ are bounded functions when $w \in R$. Put $\delta_0 = \min \delta(\mu)$ for $\mu \in [\varepsilon_0/2, \varepsilon_0]$.

The functions $\mathbf{q}_\mu(\nu w)$, $\mathbf{p}_\mu(\nu w)$ are solutions of the system

$$H_\mu' = \alpha_1\mu\nu\lambda_k(q_k^2+p_k^2)/2+\nu H_1(\nu w, \varepsilon, \mathbf{q}, \mathbf{p})$$

which transforms into (4.1) when

$$\mu\nu=\varepsilon,\ \nu H_1(w, \varepsilon, \mathbf{q}, \mathbf{p}) = \varepsilon^3 B(w/\nu, \varepsilon, \mathbf{q}, \mathbf{p}) \tag{4.7}$$

If $\nu^{-1}\in Z$, then the H_1 in (4.7) is 2π-periodic in w. Since, for any $\varepsilon\in]0, \varepsilon_0]$, there exists $N\in Z$ such that $\mu = N\varepsilon\in[\varepsilon_0/2, \varepsilon_0]$, then, putting $\nu = N^{-1}$ in (4.7), we obtain the stability condition for system (4.1)

$$\varepsilon^2\varepsilon_0 \max_{w\in[0,2\pi],\ |\mathbf{q}|=|\mathbf{p}|=1} |B(w, \varepsilon, \mathbf{q}, \mathbf{p})| \leqslant \delta_0(\lambda) \tag{4.8}$$

We have thus proved the following theorem.

Theorem 4. If no characteristic exponent of system M_1 is zero or negative, then, given sufficiently small $\varepsilon>0$, the motions of system M with ε-periodic impacts on the non-retaining constraint $q_1\geqslant 0$ are orbitally stable to a first approximation.

We will study the mechanism by which instability of the periodic solutions arises in the case when all the numbers λ_j are positive, which guarantees, in view of Theorem 2, stability of the position of equilibrium.

The derivative of function (4.1) with respect to the parameter ε is

$$dH/d\varepsilon = {}^1/_2\alpha_1\lambda_k(q_k^2+p_k^2)+3\varepsilon^2 B+o(\varepsilon^2) \tag{4.9}$$

If

$$3\varepsilon^2 \max_{w\in[0,2\pi],\ |\mathbf{q}|=|\mathbf{p}|=1} |B| < \alpha_1 \min_j \lambda_j$$

the function (4.9) is positive definite. The multiplicators of the first kind of system (4.1) which, with $\varepsilon=0$, issue from the point $\rho=1$ of the unit circle, then move along this circle counter clockwise /10/. If, with $\varepsilon=\varepsilon^*$, one multiplicator (and hence, at least two) becomes equal to -1, while the characteristic matrix then has non-simple elementary divisors, then, by the Krein-Lyubarskii theorem /10/, there exists $\varepsilon_1>\varepsilon^*$ such that, with $\varepsilon\in[\varepsilon^*, \varepsilon_1]$, the periodic motion is unstable.

There is another case when instability can arise: if the function (4.9) ceases to be positive definite as ε increases, then the multiplicator $\rho(\varepsilon)$ can start to move clockwise along the unit circle to the value $\rho=1$, which it reaches with $\varepsilon=\varepsilon^*$. Here again, the loss of stability is determined by the Krein-Lyubarskii theorem. We thus have the following theorem.

Theorem 5. If the characteristic exponents of system M' are positive, then the loss of stability of the ε-periodic motions with impacts on the non-retaining constraint for $\varepsilon>\varepsilon^*$ is accompanied by the appearance for $\varepsilon=\varepsilon^*$ of a periodic $(\rho=1)$ or antiperiodic $(\rho=-1)$ solution of the equations in variations.

Example. A particle which moves in a vertical plane above a smooth curve $y=f(x)$ is in stable equilibrium at the critical point x° if $f''(x^\circ)>0$; then $\lambda=[f''(x^\circ)]^{1/2}$. The motions with periodic impacts are described by the relations

$$x=x^\circ,\ \ y={}^1/_2 g\,|t|\,({}^1/_2\varepsilon-|t|)\quad \text{for}\quad |t|\leqslant {}^1/_2\varepsilon,\ \ y(t+\varepsilon)=y(t)$$

while the height h to which the particle jumps is $g\varepsilon^2/8$. If the curve is replaced to a first approximation by a parabola $y={}^1/_2 f''(x^\circ)(x-x^\circ)^2$, we can choose h in such a way that the system has antiperiodic motion: for this, the particle must strike the parabola at a right angle. The velocity vector $(v, -{}^1/_4 g\varepsilon)$ must be orthogonal to the tangent vector to the parabola $(1, {}^1/_2 f''(x^\circ)v\varepsilon)$, whence we have

$$1={}^1/_8 f''(x^\circ)g\varepsilon^2 = 2f''(x^\circ)h$$

which is the same as the stability domain boundary $0<f''(x^\circ)h<{}^1/_2$, see /11/.

If the λ_k have different signs, the loss of stability is not necessarily linked with the birth of periodic or antiperiodic solutions; resonance of composite type, when $r\neq s$ in (4.2), can also lead to instability.

5. *Stability in the case $n=2$.* In the case $n=2$ we study the orbital stability of periodic motions in the strict non-linear statement.

We showed in Sect.4 that, if the equilibrium position of system M_1 is stable, then, for sufficiently small values of ε, the Hamiltonian of the disturbed motion is

$$r^* = {}^1/_2\lambda(q_2^2+p_2^2)+r_1(E, w, q_2, p_2) \tag{5.1}$$

where the function r_1 is continuous and 2π-periodic in w and depends smoothly on the energy constant E and on q_2, p_2.

The orbital stability of periodic motions can thus be determined on the basis of the Arnol'd-Moser theorem /8, 12/. If none of $\lambda, 2\lambda, \ldots, 2l\lambda$ is an integer, we can reduce system (5.1) by non-linear real canonical transformation to the normal form

$$r^* = \lambda r_2 + c_2 r_2^2 + \ldots + c_l r_2^l + r_{2l+1}, \quad 2r_2 = q_2^2 + p_2^2 \tag{5.2}$$

In (5.2), r_{2l+1} is a smooth function in q_2, p_2 whose order of smallness is at least $2l+1$, and which is continuous and 2π-periodic in w. If at least one coefficient c_j ($j = 2, \ldots, l$) is non-zero, then, by the Arnol'd-Moser theorem, the trivial solution of system (5.1) is stable. Some resonance cases, when $s\lambda$ is an integer for $s \in Z^+$, are considered in /3, 7/.

6. Stability in the multidimensional case. The present level of development of stability theory prevents us from treating the case $n \geqslant 3$ as fully as the case $n = 2$. Strict results on Lyapunov stability or instability of the equilibrium position of a system, established as a result of the reaction of the non-retaining constraint, are exhausted by the cases covered by Theorems 1 and 2. For applications, however, it may be useful to consider stability for the majority of initial conditions.

According to the Kolmogorov-Arnol'd-Moser theory, stability for the majority of initial conditions is inherent in the equilibrium positions of Hamiltonian systems when the Hamiltonian of the disturbed motion is reasonably smooth and when certain conditions for non-degeneracy of the normal form hold /8, 13/. A feature of the present case is that H is not differentiable with respect to q_1 for $q_1 = 0$. We shall show that this does not affect the main conclusion of KAM theory on the existence of invariant tori which ensure stability for most initial conditions.

Using the results of Sect.4, we write the Hamiltonian function in the neighbourhood of the equilibrium position as

$$H = \alpha I_1^{2/3} + {}^1/_2\lambda_k (q_k^2 + p_k^2) + H_1(\mathbf{q}, \mathbf{p}, I_1, w) \tag{6.1}$$

where $\partial H_1/\partial \mathbf{q} = \partial H_1/\partial \mathbf{p} = 0$ for $\mathbf{q} = \mathbf{p} = 0$, $\partial^2 H_1/\partial \mathbf{q}^2 = \partial^2 H_1/\partial \mathbf{p}^2 = \partial^2 H_1/\partial \mathbf{q}\lambda\mathbf{p} = 0$ for $\mathbf{q} = \mathbf{p} = I_1 = 0$; the function H_1 is smooth (is differentiable a sufficient number of times for application of the methods of KAM theory) with respect to $\mathbf{q}, \mathbf{p}, I_1$, and is continuous and 2π-periodic in w.

We assume that resonance relations of up to and including the fourth order do not exist between the characteristic exponents, i.e., that we have no equation of the type

$$\sum_{j=2}^{n} m_j \lambda_j = 0, \quad m_j \in Z, \quad \sum_{j=2}^{n} |m_j| \leqslant 4 \tag{6.2}$$

Putting $I_1 = I_0 + r_1$, $I_0 = (E/\alpha)^{3/2}$, we perform iso-energetic reduction in (6.1), taking $H = E$ and w as the new independent variable. The results is a system with $n-1$ degrees of freedom:

$$r_1 = {}^1/_2\varepsilon\lambda_k^* (q_k^2 + p_k^2) + O(\varepsilon^2), \quad \varepsilon \sim I_0^{1/3}, \quad \lambda_k^* = \lambda_k^*(\varepsilon), \quad \lambda_k^*(0) = \lambda_k \tag{6.3}$$

where terms of at least the third order in $\mathbf{q}, \mathbf{p}$ are not written.

Since resonance of type (6.2) do not exist for the λ_j, they also do not exist for the λ_j^* for sufficiently small ε. It then follows that, for small ε, there are no resonances of the more general type

$$\varepsilon \sum_{j=2}^{n} \lambda_j^* m_j \in Z, \quad m_j \in Z, \quad \sum_{j=2}^{n} |m_j| \leqslant 4 \tag{6.4}$$

which are important when the Hamiltonian depends explicitly on the independent variable.

As a result of normalization, system (6.3) up to fourth order terms, becomes

$$r_1 = \varepsilon\lambda_k^* r_k + {}^1/_2 c_{km} r_k r_m + O(\varepsilon^2), \quad 2r_k = q_k^2 + p_k^2 \tag{6.5}$$

where the coefficients c_{km} vanish along with ε, and the unwritten terms have order at least $^5/_2$ in r_k, and represent the disturbed part of the system, which latter is integrable when this part is absent. The difference between system (6.5) and the cases usually treated in KAM theory is that the disturbances are not smooth with respect to the independent variable w. But we showed in Sect.5 that, in the case $n = 2$, this fact does not play an important role (see also /14/, the note on 192, 194). Even in the case $n \geqslant 3$ non-differentiability with respect to w is not important, as is shown by the following theorem /15/.

Theorem 6. If the non-degeneracy condition $\det\|c_{km}\| \neq 0$ holds (as applied to system (6.1), this is the condition for iso-energetic non-degeneracy), and ε, r_k are sufficiently small, most of the non-resonant tori of the system with Hamilton function

$$H_0 = \varepsilon\lambda_k^* r_k + {}^1/_2 c_{km} r_k r_m$$

do not vanish, and only a few are deformed, so that, in the phase space of system (6.5), there are also invariant tori, filled everywhere densely by phase curves which wrap round them conditionally periodically with $n-1$ frequencies. These invariants tori form a majority in the sense that the Lebesgue measure of the complementary union is small along with the disturbances. These latter are assumed to belong to class C_s, $s > 2n$ /15/.

Proof. If the dependence on w in system (6.3) is smooth (class C_s), we could use Arnol'd's /13/ or Moser's /8/ proof as applied to the initial system (6.1). It is still possible to use these proofs in the present case, by extending them to systems with an unsmooth dependence of the Hamiltonian on time (here, on the independent variable w).

Let $S = S(w, \mathbf{r}', \boldsymbol{\varphi})$ be the generating function of the normalizing transformation $\mathbf{r}, \boldsymbol{\varphi} \to \mathbf{r}', \boldsymbol{\varphi}'$, so that $\mathbf{r} = \mathbf{r}' + \partial S/\partial \boldsymbol{\varphi}$, $\boldsymbol{\varphi}' = \boldsymbol{\varphi} + \partial S/\partial \mathbf{r}'$.

The new Hamiltonian is $r_1' = r_1(\mathbf{r}' + \partial S/\partial\boldsymbol{\varphi}, \boldsymbol{\varphi}' - \partial S/\partial \mathbf{r}, w) + \partial S/\partial w$. Noting that r_1 and S are periodic with respect to the angular variables, they can be expanded in infinite sums

$$r_1 = \sum h_{\mathbf{u}}(\mathbf{r}, w)\, e^{i(\mathbf{u}, \boldsymbol{\varphi})}, \quad S = \sum S_{\mathbf{u}}(\mathbf{r}, w)\, e^{i(\mathbf{u}, \boldsymbol{\varphi})}$$
$$\mathbf{u} = (u_2, \ldots, u_n), \quad (\mathbf{u}, \boldsymbol{\varphi}) = u_2\varphi_2 + \cdots + u_n\varphi_n$$

To nullify in the normal form the terms that depend on $\boldsymbol{\varphi}$, the coefficients S_k must satisfy the equations /3/

$$\partial S_{\mathbf{u}}/\partial w + i\varkappa_{\mathbf{u}} S_{\mathbf{u}} + h_{\mathbf{u}} = 0, \quad \varkappa_{\mathbf{u}} = (\boldsymbol{\omega}, \mathbf{u}) \tag{6.6}$$

where the $\omega_k = \partial H_0/\partial r_k$ are evaluated on the invariant torus $r_k = \text{const}$. Eq.(6.6) has a unique 2π-periodic solution, given by relations similar to (3.9). In the relations, the denominator vanishes if $\varkappa_k$ is an integer, i.e., in the case of resonances of type (6.4).

Estimation of $|S_k|$ is based on an arithmetic lemma which generalizes the well-known assertion of /8/ concerning resonances of type (6.2).

Lemma. For almost all $\boldsymbol{\omega} \in \mathbb{R}^n$ we have

$$|1 - e^{-2\pi i(\boldsymbol{\omega}, \mathbf{u})}| \geqslant C(\boldsymbol{\omega})\, |\mathbf{u}|^{-\nu}, \quad \nu \geqslant n+1 \tag{6.7}$$

Proof. Since

$$|1 - e^{-2\pi i(\boldsymbol{\omega}, \mathbf{u})}| = 2\,|\sin \pi(\boldsymbol{\omega}, \mathbf{u})|$$

the estimated quantity is close to zero when the scalar product $(\boldsymbol{\omega}, \mathbf{u})$ is close to an integer. It is well-known that, for almost all $(n+1)$-dimensional vectors $\boldsymbol{\omega}^* = \{\boldsymbol{\omega}, 1\}$,

$$|(\mathbf{u}^*, \boldsymbol{\omega}^*)| = |(\mathbf{u}, \boldsymbol{\omega}) + u_{n+1}| \geqslant C_1(\boldsymbol{\omega})\,(|\mathbf{u}| + |u_{n+1}|)^{-\nu} \tag{6.8}$$

If $|u_{n+1}| > 2\,|\mathbf{u}|\,|\boldsymbol{\omega}|$, then $|\mathbf{u}|\,|\boldsymbol{\omega}| < |(\mathbf{u}, \boldsymbol{\omega}) + u_{n+1}|$, while otherwise it follows from (6.8) that

$$|(\mathbf{u}, \boldsymbol{\omega}) + u_{n+1}| \geqslant C_2(\boldsymbol{\omega})\,|\mathbf{u}|^{-\nu}, \quad C_2 = C_1(1 + 2\,|\boldsymbol{\omega}|)^{-\nu}$$

Putting $C_3(\boldsymbol{\omega}) = \min\{2\,|\boldsymbol{\omega}|, C_2(\boldsymbol{\omega})\}$, we obtain $|(\mathbf{u}, \boldsymbol{\omega}) + u_{n+1}| \geqslant C_3(\boldsymbol{\omega})\,|\mathbf{u}|^{-\nu}$.

Since $|\sin \pi x| \geqslant \min {}^1/_2\pi\,|x - N|$ $(N \in Z)$ for all $x \in R$, we finally have

$$|1 - e^{-2\pi i(\boldsymbol{\omega}, \mathbf{u})}| = 2\,|\sin \pi(\boldsymbol{\omega}, \mathbf{u})| = 2\,|\sin[\pi(\boldsymbol{\omega}, \mathbf{u}) + \pi u_{n+1}]| \geqslant \pi \min|(\boldsymbol{\omega}, \mathbf{u}) + u_{n+1}| \geqslant \pi C_3(\boldsymbol{\omega})\,|\mathbf{u}|^{-\nu}$$

which proves the lemma.

By (6.7), we have

$$|S_{\mathbf{u}}| < C_4(\boldsymbol{\omega})\,|h_{\mathbf{u}}|\,|\mathbf{u}|^{-\nu} \tag{6.9}$$

which has the same form regardless of whether the Hamilton function is smooth with respect to the independent variable w. Satisfaction of (6.9) ensures that the iterative process is convergent when constructing the invariant tori of the disturbed system, so that the rest of the proof of Theorem 6 is the same as in the smooth case /8, 13/.

REFERENCES

1. SUSLOV G.K., Theoretical Mechanics, Gostekhizdat, Moscow-Leningrad, 1946.
2. IVANOV A.P., On stability in systems with non-retaining constraints, PMM, 48, 5, 1984.
3. MARKEYEV A.P., Points of Libration in Celestial Mechanics and Cosmo-dynamics, Nauka, Moscow, 1978.
4. IVANOV A.P. and MARKEYEV A.P., On the dynamics of systems with one-sided constraints, PMM, 4, 1984.
5. KAMKE E., Guide to Ordinary Differential Equations, Edwards Bros., Ann Arbor, Michigan, 1966.

6. LYAPUNOV A.M., The General Problem of the Stability of Motion Collected Papers, Izd-vo Akad. Nauk SSSR, Moscow-Leningrad, 1956.
7. ZIGEL K.L., Lectures on Celestial Mechanics, IIL, Moscow, 1959.
8. MOSER J., Lectures on Hamiltonian Systems, AMS, 1969.
9. HANNING R. and ROSSI H., Analytic Functions of Several Complex Variables, Mir, Moscow, 1969.
10. YAKUBOVICH V.A. and STARZHINSKII V.M., Linear Differential Equations with Periodic Coefficients and their Applications, Nauka, Moscow, 1972.
11. KOBRINSKII A.A. and KOBRINSKII A.E., Two-dimensional Vibro-impact Systems, Nauka, Moscow, 1981.
12. ARNOL'D V.I., On the stability of the equilibrium position of a Hamiltonian system of ordinary differential equations in the general elliptic case, Dokl. Akad. Nauk SSSR, 137, 2, 1961.
13. ARNOL'D V.I., Proof of Kolmogorov's theorem on the conservation of conditionally periodic motions with a small change of the Hamilton function, Usp. Mat. Nauk, 18, 5, 1963.
14. LIKHTENBERG A. and LIEBERMANN M., Regular and Stochastic Dynamics, Mir, Moscow, 1984.
15. KOLMOGOROV A.N., On conservation of conditionally periodic motions with a small change of the Hamilton function, Dokl. Akad. Nauk SSSR, 98, 4, 1954.

Translated by D.E.B.

PMM U.S.S.R., Vol.53, No.4, pp.424-429, 1989
Printed in Great Britain

0021-8928/89 $10.00+0.00

ASYMPTOTIC MOTIONS OF MECHANICAL SYSTEMS WITH NON-HOLONOMIC CONSTRAINTS*

G.M. VINNER

The motions of mechanical systems with non-holonomic constraints close to critical points of the potential are considered. The stability of the equilibrium positions was first treated by Whittaker /1/. A theorem is given which includes earlier results /2/ as a special case, and which enables asymptotic motions to be found for new classes of potentials. Sufficient conditions are found for the equilibrium to be unstable when not all the frequencies of small oscillations vanish. Similar studies were made in /3-6/ for systems without constraints.

The hypothesis can be advanced that a critical point of the potential energy is an unstable equilibrium of a mechanical system with non-holonomic constraints (linear in the velocity) when zero is not a minimum of the function V^*.

Here, the origin is the equilibrium position in question, and the asterisk denotes contraction of the potential energy V to the subspace, orthogonal to all the constraints at zero.

This hypothesis is proved below for the case when the MacLaurin expansion of V^* is $V^* = V_2^* + V_k^* + V_{k+1}^* + \ldots$, where $V_2^* + V_k^*$ can take negative values infinitesimally close to zero (V_j^* is a homogeneous form of degree j).

This situation when $V_2^* \geqslant 0$ and $V_k^* \geqslant 0$ is not considered. Also, to determine the absence of a minimum, higher powers must be taken into account.

1. The rigorous statement of the problem, and the result. We consider a mechanical system with configuration space L, which can be regarded as the standard R^n, since all our constructions are performed in an infinitesimally small neighbourhood of zero. Let the generalized coordinates be $\xi = (\xi^1, \ldots, \xi^n)^T \in L$. The Lagrangian of the system can be written as $K(\xi, \xi^{\cdot}) - V(\xi)$, where K is the kinetic energy, quadratic in the velocity, and V is the

**Prikl. Matem. Mekhan.*, 53, 4, 549-555, 1989